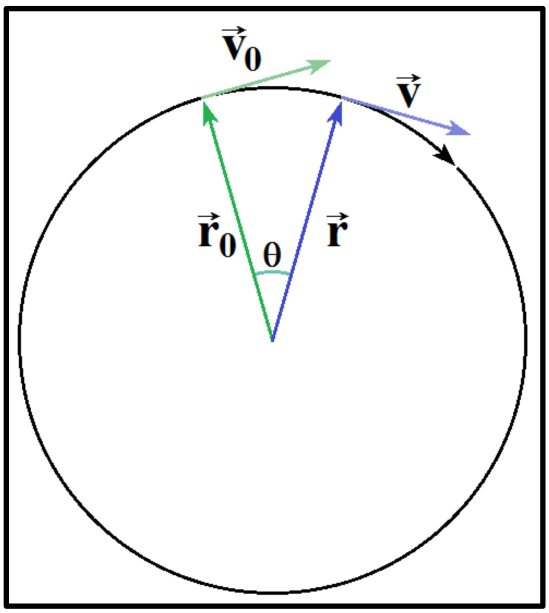
Part 6: Uniform Circular Motion

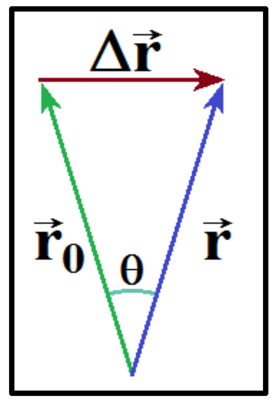
Physics **(Giancoli): Chapter 5**
College Physics **(Openstax): Chapter 6**

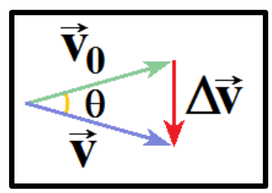
Uniform Circular Motion

- **Uniform Circular Motion** is moving in a circular path at a constant speed.
- **The Period (T)** is the time needed to complete one full cycle (once around the circle).
- **The Frequency (f)** is the number of cycles complete divided by the time interval.

$$\mathbf{T = \frac{1}{f} \qquad v = \frac{2\pi r}{T} = 2\pi r f}$$







The distance triangle (upper right) and the velocity triangle are similar triangles (identical interior angles). The ratios of corresponding sides of similar triangles are equal.

$$\frac{\Delta r}{r} = \frac{\Delta v}{v}$$

For small time intervals ($\Delta t \to 0$), the angle theta is very small. This allows us to approximate Δr.

$$\Delta r \approx v \cdot \Delta t$$

Plugging into the previous equation leads to our result.

$$\frac{v\Delta t}{r} = \frac{\Delta v}{v} \qquad \frac{v^2 \Delta t}{r} = \Delta v \qquad \frac{v^2}{r} = \frac{\Delta v}{\Delta t}$$

For small time intervals ($\Delta t \to 0$), $a = \Delta v/\Delta t$. In this case, the acceleration is referred to as "centripetal acceleration (a_c) or radial acceleration (a_r). Centripetal acceleration always points in the direction of Δv, radially inward towards the center of the circle.

$$a_c = \frac{v^2}{r}$$

As the centripetal acceleration a_c is perpendicular to velocity with no component in the direction of the velocity, it only changes the direction and not the magnitude of the velocity.

Example: Computer-controlled display screens provide drivers in the Indianapolis 500 with a variety of information about how their cars are performing. For instance, as a car is going through a turn, a speed of 221 mi/h (98.8m/s) and a centripetal acceleration of 3.00g are displayed. Determine the radius of the turn.

$$3.00g = (3.00)\left(9.80\frac{m}{s^2}\right) = 29.4\ \frac{m}{s^2}$$

$$a_c = \frac{v^2}{r} \qquad a_c \cdot r = v^2 \qquad r = \frac{v^2}{a_c} = \frac{\left(98.8\frac{m}{s}\right)^2}{29.4\frac{m}{s^2}} = 332\ m$$

Centripetal Forces

- The **Centripetal Force** is the force that gives rise to the centripetal acceleration that causes an object to move in a curved path.

$$F_C = ma_C = \frac{mv^2}{r}$$

- The centripetal force is not a new force. Any of the other forces we've studied (or a combination) may act as the centripetal force.
- The centripetal force is always directed at the center of the circle, perpendicular to the velocity.

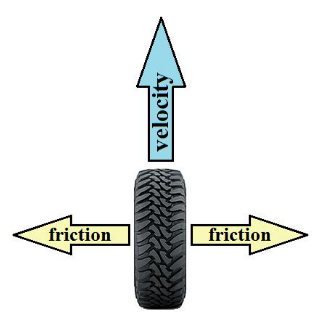
Example: In a skating stunt known as "crack-the-whip", a number of skaters hold hands and form a straight line. They skate so that the line rotates around the skater at one end who acts as a pivot. The skater farthest out has a mass of 80.0kg and is 6.10m from the pivot. He is skating at a speed of 6.80m/s. Determine the magnitude of the centripetal force that acts on him.

$$F_c = \frac{mv^2}{r} = \frac{(80.0\ kg)\left(6.80\frac{m}{s}\right)^2}{6.10\ m} = 606\text{ N}$$

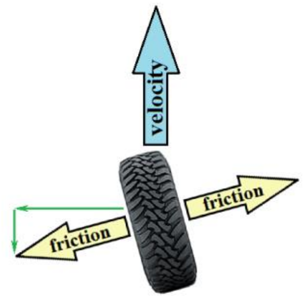
Example: Compare the maximum speeds at which a car can safely negotiate an unbanked turn (radius = 50.0m) in dry weather (μ_s = 0.900) and icy weather (μ_s = 0.100)

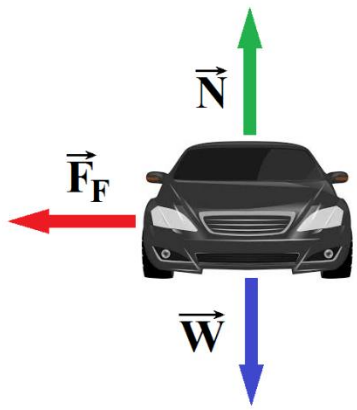
Note: Rolling (not sliding) wheels are dependent upon μ_s not μ_k

The bottom of a rolling tire is stationary on the road's surface. This is evident when it rolls through water as the tread of the tire is visible in the water trail it leaves behind. This rolling motion prevents any friction forces from being felt in the direction the tire rolls (unless brakes are applied disrupting the rolling motion). Consequently, a rolling tire can only feel friction forces parallel to the tires axle.



When are car turns, the tires can only exert a friction force on one side or the other. One side would have a friction component in the direction of motion (which is not allowed). The friction force is thus felt on the other side with two components. One component is perpendicular to the motion, causing the car to turn. The other is opposite the velocity, causing the car to decelerate.





There are three forces acting on the car: the weight, the normal force, and the friction force.

Step 1: Determine which force is acting as F_C. Remember this force points to the center of the circle the object moves in. In this case, the friction force is acting as the centripetal force.

Step 2: Solve for F_C: $F_C = F_F = \mu_s N = \mu_s mg$

Step 3: Set F_C equal to $\frac{mv^2}{r}$:

$$\frac{mv^2}{r} = \mu_s mg \quad \frac{v^2}{r} = \mu_s g \quad v^2 = \mu_s gr \quad v = \sqrt{\mu_s gr}$$

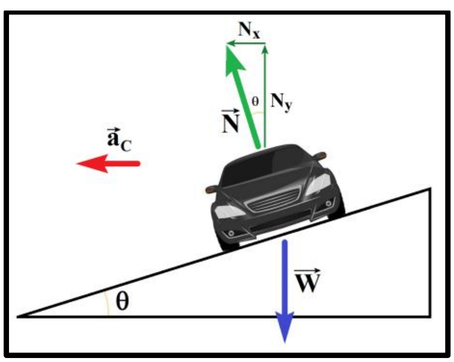
For dry weather ($\mu_S = 0.900$): $v = \sqrt{\mu_s gr} = \sqrt{(0.900)\left(9.80\frac{m}{s^2}\right)(50.0\text{ m})} = 21.0\frac{m}{s}$ (47 mph)

For icy weather ($\mu_S = 0.100$): $v = \sqrt{\mu_s gr} = \sqrt{(0.100)\left(9.80\frac{m}{s^2}\right)(50.0\text{ m})} = 7.00\frac{m}{s}$ (15.7 mph)

Note: Anti-lock brakes keep a car from skidding. Skidding cars are affected by the coefficient of kinetic friction, which is typically smaller than the coefficient of static friction. A non-skidding car affected by the larger static coefficient will stop faster.

Frictionless Banked Curve

There are only two forces on a frictionless banked curve (weight and the normal force). When combined these forces result in an acceleration that is inward.



Step 1: Determine which force is acting as F_C.

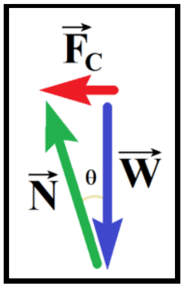
What force points to the center? $N_x = F_C$.

Step 2: Solve for F_C: $\text{Tan}\,\theta = \frac{N_x}{N_y} = \frac{F_C}{W} = \frac{F_C}{mg}$

$$F_C = mg\text{Tan}\,\theta$$

Step 3: Set $F_C = mv^2/r$: $F_C = mg\text{Tan}\,\theta = \frac{mv^2}{r}$

$$g\text{Tan}\,\theta = \frac{v^2}{r} \qquad v^2 = rg\text{Tan}\,\theta \qquad v = \sqrt{rg\text{Tan}\,\theta}$$



Alternatively, we could have made a triangle from a simple vector equation:

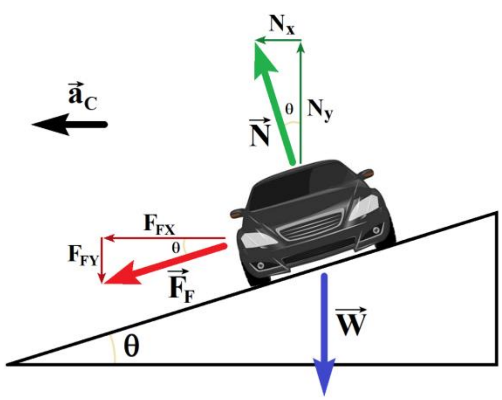
$$\vec{W} + \vec{N} = \vec{F}_C$$

$$\text{Tan}\,\theta = \frac{F_C}{W} = \frac{F_C}{mg}$$

Banked Curve with Friction

There are three forces on a frictionless banked curve (weight, the normal force, and friction). When combined these forces result in an acceleration that is inward.

The direction of the friction force is dependent upon circumstances. A slow moving car would tend to slide down the incline (opposing this, the friction would point up the incline). Alternatively, a fast moving car would tend to skid outward and up the incline. In this latter case the friction would point down the incline. As we will be considering maximum velocities, we will have the friction pointing down the slope.



Step 1: Determine which force(s) is acting as F_C.

What force(s) points to the center? $F_C = N_x + F_{Fx}$.

Step 2: Solve for F_C: start with x-components

$$F_c = N_x + F_{Fx} = N\text{Sin}\,\theta + F_F\text{Cos}\,\theta$$

$$F_c = N\text{Sin}\,\theta + \mu_S N\text{Cos}\,\theta = (\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta)N$$

Need N from y-components: $N_y = W + F_{Fy}$

$$N\text{Cos}\,\theta = mg + F_F\text{Sin}\,\theta = mg + \mu_S N\text{Sin}\,\theta$$

$$N\text{Cos}\,\theta - \mu_S N\text{Sin}\,\theta = mg \qquad (\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta)N = mg \qquad N = \frac{mg}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}$$

Plug N (from y-components) back in to the equation for F_C (from the x-components):

$$F_c = (\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta)N = (\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta)\left(\frac{mg}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}\right)$$

$$F_c = mg\left(\frac{\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}\right)$$

Step 3: Set $F_C = mv^2/r$: $\quad F_c = mg\left(\frac{\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}\right) = \frac{mv^2}{r} \qquad g\left(\frac{\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}\right) = \frac{v^2}{r}$

$$v^2 = rg\left(\frac{\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}\right) \qquad v = \sqrt{rg\left(\frac{\text{Sin}\,\theta + \mu_S\text{Cos}\,\theta}{\text{Cos}\,\theta - \mu_S\text{Sin}\,\theta}\right)}$$

Earlier we solved an unbanked turn with friction and got the maximum velocity as $v = \sqrt{\mu_s gr}$.

This should be equivalent to the banked turn with friction if we let the banking angle be zero.

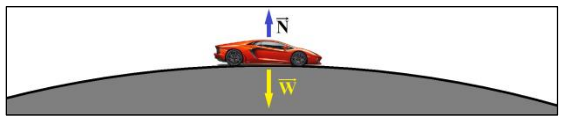
$$v = \sqrt{rg\left(\frac{\text{Sin } 0 + \mu_S \text{Cos } 0}{\text{Cos } 0 - \mu_S \text{Sin } 0}\right)} = \sqrt{rg\left(\frac{0 + \mu_S \cdot 1}{1 - \mu_S \cdot 0}\right)} = \sqrt{\mu_S gr}$$

Earlier we solved a banked turn without friction and got the velocity as $v = \sqrt{rg\text{Tan } \theta}$

This should be equivalent to the banked turn with friction if we let μ_S be zero.

$$v = \sqrt{rg\left(\frac{\text{Sin } \theta + 0 \cdot \text{Cos } \theta}{\text{Cos } \theta - 0 \cdot \text{Sin } \theta}\right)} = \sqrt{rg\left(\frac{\text{Sin } \theta}{\text{Cos } \theta}\right)} = \sqrt{rg\text{Tan } \theta}$$

Example: A car drives over a hill. At the crest, the radius of curvature is 50.0 m. What is the maximum speed the car can have and still keep its tires on the road?



Step 1: Which force(s) is acting as F_C?

The centripetal force must be directed vertically downward (towards the center of the circle). As there are two vertical forces acting on the car, these must combine to act as the centripetal force.

$$F_C = W - N$$

Step 2: Solve for F_C: $F_C = W - N = mg - N$

To determine the maximum velocity, we need the maximum centripetal force. W=mg is fixed in value. The normal force can take any positive value. The maximum value of F_C comes when N=0.

$$F_C = mg$$

Step 3: Set $F_C = mv^2/r$: $F_C = mg = \frac{mv^2}{r}$ $g = \frac{v^2}{r}$ $v^2 = gr$

$$v = \sqrt{gr} = \sqrt{\left(9.80\frac{m}{s^2}\right)(50.0 \text{ m})} = 22.1\frac{m}{s}$$

The Forces of Nature

- The **Gravitational Force** creates forces between massive objects. It is responsible for pulling objects to the surface of the Earth, causing planets to orbit the sun, and causing stars to orbit the centers of galaxies.
- The **Electromagnetic Force** creates forces between electric charges. It allows us to generate and use electrical power, transmit signals across vast distances, and use a multitude of convenient devices.
- The **Weak Nuclear Force** is most noticeable in radioactive decays, but also plays a role in nuclear fission (as used in nuclear power plants and nuclear bombs).

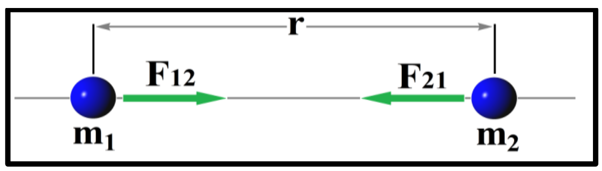
- The **Strong Nuclear Force** binds quarks into hadrons (such as protons and neutrons). In addition, it holds atomic nuclei together.

Newton's Law of Gravity

- Newton's Law of Gravity determines the magnitude of the gravitational force (F) between two objects.

$$F = G\frac{m_1 m_2}{r^2}$$

  - F is the force on both massive objects (equal and opposite)
  - G is the gravitational constant $\left(G = 6.673 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}\right)$
  - m_1 and m_2 are the masses of the two objects
  - r is the distance separating the two object's centers of mass.
- This force is attractive, and is directed along the line connecting the two object's centers of mass.



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- The gravitational acceleration (g) is caused by the Earth's gravitational pull and can be directly calculated from Newton's law of gravity, given the mass of the Earth ($M_E = 5.98 \times 10^{24}$ kg), the radius of the earth ($R_E = 6.38 \times 10^6$ m) and the gravitational constant ($G = 6.673 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}$).

$$W = G\frac{M_E m}{R_E^2} = m\left[\frac{GM_E}{R_E^2}\right] = mg$$

$$g = \frac{GM_E}{R_E^2} = \frac{\left(6.673 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}\right)\left(5.98 \times 10^{24}\ kg\right)}{(6.38 \times 10^6\ m)^2} = 9.8035\,\frac{m}{s^2}$$

Satellites in Circular Orbits

- For orbiting satellites, gravitational attraction acts as the centripetal force. Let's let the mass of a satellite be m_s and the mass of the Earth be M_E.

$$F_C = F_g \qquad \frac{m_S v^2}{r} = G\frac{m_S M_E}{r^2} \qquad \frac{v^2}{r} = G\frac{M_E}{r^2} \qquad v^2 = G\frac{M_E}{r} \qquad v = \sqrt{G\frac{M_E}{r}}$$

- Each orbital radius can only be held by an object with a specific velocity, and that velocity is independent of that object's mass:

$$v = \sqrt{G\frac{M_E}{r}}$$

Example: The moon orbits the Earth at a distance of 3.85 x 10^8 m. The Earth's mass is 5.98 x 10^{24} kg. Find the period for the moon's motion around the Earth (in days).

$$v = \sqrt{G\frac{M_E}{r}} = \sqrt{\left(6.673 \times 10^{-11}\frac{Nm^2}{kg^2}\right)\frac{(5.98 \times 10^{24}\ kg)}{(3.85 \times 10^{8}\ m)}} = 1018.077\frac{m}{s}$$

$$v = \frac{2\pi r}{T} \qquad T = \frac{2\pi r}{v} = \frac{2\pi(3.85 \times 10^8\ m)}{1018.077\frac{m}{s}} = 2{,}376{,}073\ s\left(\frac{1\ hour}{3600\ s}\right)\left(\frac{1\ day}{24\ hours}\right) = 27.5\ days$$

Example: Objects orbiting the center of galaxy M87 have been found to have an orbiting speed of 7.5 x 10^5 m/s for matter orbiting at a distance of 5.7 x 10^{17} m from the center. Find the mass (M) of the object located at the center.

$$v = \sqrt{G\frac{M_{obj}}{r}} \qquad v^2 = G\frac{M_{obj}}{r} \qquad v^2 r = GM_{obj}$$

$$M_{obj} = \frac{v^2 r}{G} = \frac{\left(7.5 \text{ x } 10^5\ \frac{m}{s}\right)^2 (5.7 \text{ x } 10^{17}\ m)}{6.673 \times 10^{-11}\frac{Nm^2}{kg^2}} = 4.8 \times 10^{39}\ kg$$

As the mass of our sun is 2.0 $\times 10^{30}$ kg, the object at the center of M87 is 2.4 billion solar masses. Due to the limited volume available for this amount of mass it was concluded that the object at the center of M87 was a black hole.

Note: Because of the equivalence of inertial mass and gravitational mass, Einstein concluded that there should be no difference between gravity and an accelerating reference frame. This is known as "the equivalence principle", and it led him to conclude that light should bend in a gravitational field even though it has no mass.

Artificial Gravity

- For accelerating reference frames we can account for the acceleration by adding the effects in with gravity, creating "apparent gravity". We used a_F as the acceleration of the reference frame.

$$\vec{g}_{app} = \vec{g} - \vec{a}_F$$

- For an object in orbit around the Earth, the gravitational force acts as the centripetal acceleration. Under these conditions the apparent gravity is zero.

$$\vec{a}_F = \vec{g} \qquad \vec{g}_{app} = \vec{g} - \vec{g} = 0$$

- Astronauts spending long periods in space suffer debilitating conditions due to the lack of gravity, including loss of bone and muscle mass. Artificial gravity provides a means of eliminating these effects, allowing the life forms of Earth to spend an indefinite amount of time in space.

- By making the object in orbit rotate, we can introduce an additional centripetal force that can be used to create artificial gravity. This gravity is equal to a_C, but pointing radially outward.

$$\vec{a}_F = \vec{g} + \vec{a}_C \qquad \vec{g}_{app} = \vec{g} - \vec{a}_F = \vec{g} - (\vec{g} + \vec{a}_C) = -\vec{a}_C$$

Example: The outer wall of a toroidal (donut-shaped) space station is 50.0 m from its central axis. If the toroid rotates uniformly to create a gravity equivalent to Earth's surface, what is the speed of the outer wall?

$$a_c = g = \frac{v^2}{r} \qquad \mathrm{v} = \sqrt{\mathrm{gr}} = \sqrt{\left(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(50.0\ \mathrm{m})} = 22.1\,\frac{\mathrm{m}}{\mathrm{s}}$$